# A SHORT REMARK ON CONTROL SYSTEMS

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ABSTRACT. Souza and Tozatti [7] introduce the notions of prolongations and prolongational limit sets on control systems. In this article, we prove the upper semicontinuity of first positive prolongations and first positive prolongational limit sets on control systems.

## 1. Introduction

In the present paper, we study continuity of first prolongations and first prolongational limit sets on control systems. The notions for original dynamical systems are appeared into detail, see [1]. As for control systems, Souza and Tozatti [7] introduced the notions on the systems and they investigated dynamical properties about the notions.

To explain control systems, we first let $M$ a connected $d$-dimensional smooth manifold. For a subset $U$ of $\mathbb{R}^n$, denote by $\mathcal{U}_{pc}$ the set of all piecewise constant maps from $\mathbb{R}$ to $U$. Let $X : M \times U \to TM$ be a continuous map such that $X(\cdot, u) : M \to TM$ is a complete vector field on $M$ for every $u \in U$.

Now, we consider the control system

$$(\textstyle\sum) \qquad \dot{x}(t) = X(x(t), u(t)), \ \ u \in \mathcal{U}_{pc}$$

on $M$. For each $x \in M$ and $u \in \mathcal{U}_{pc}$, the preceding equation has a unique solution $\phi(t, x, u)$, $t \in \mathbb{R}$ with $\phi(0, x, u) = x$. For more results about control systems, we refer the reader to [2, 3, 4, 5].

It is well-known that if the subset $U$ of $\mathbb{R}^n$, called the *control range* of the system, is a compact convex subset of $\mathbb{R}^n$, then the closure $\mathcal{U} := \mathrm{cls}(\mathcal{U}_{pc})$ of $\mathcal{U}_{pc}$ with respect to the weak*-topology is a compact Hausdorff space as a subspace of $L^\infty(\mathbb{R}, \mathbb{R}^n)$ and the map $\phi : \mathbb{R} \times M \times \mathcal{U}_{pc} \to M$ can be extended to continuous map $\phi : \mathbb{R} \times M \times \mathcal{U} \to M$ (see [5] for details). Here, $\mathrm{cls}(\mathcal{U}_{pc})$ is the closure of $\mathcal{U}_{pc}$.

From now on, we assume that each control range $U$ is compact and convex.

For a control system, we define the *trajectory mapping* of the system given by

$$\mathcal{S} := \{e^{t_n Y_n} e^{t_{n-1} Y_{n-1}} \cdots e^{t_0 Y_0} : Y_j \in F, \, t_j \geq 0, \, n \in \mathbb{N}\},$$

where $F := \{X(\cdot, u) : u \in U\}$ is the set of vector fields. In other words, for every $x \in M$, we can put $\mathcal{S}(x) := \{\phi(t, x, u) : t \geq 0, \, u \in \mathcal{U}_{pc}\}$ and $\mathcal{S}^{-1}(x) := \{\phi(t, x, u) : t \leq 0, \, u \in \mathcal{U}_{pc}\}$. The sets $\mathcal{S}(x)$ and $\mathcal{S}^{-1}(x)$ are said to be *positive orbit* and *negative orbit* of $x$, respectively.

For $t \geq 0$, we define the sets

$$\mathcal{S}_{\leq t} := \left\{ e^{t_n Y_n} e^{t_{n-1} Y_{n-1}} \cdots e^{t_0 Y_0} : Y_j \in F, \, t_j \geq 0, \, \sum_{j=0}^{n} t_j \leq t, \, n \in \mathbb{N} \right\},$$

$$\mathcal{S}_{\geq t} := \left\{ e^{t_n Y_n} e^{t_{n-1} Y_{n-1}} \cdots e^{t_0 Y_0} : Y_j \in F, \, t_j \geq 0, \, \sum_{j=0}^{n} t_j \geq t, \, n \in \mathbb{N} \right\}.$$

Observe that, for $t \in \mathbb{R}^+$, $\mathcal{S} = \mathcal{S}_{\geq 0} = \mathcal{S}_{\leq t} \cup \mathcal{S}_{\geq t}$. We set $\mathcal{F} := \{\mathcal{S}_{\geq t} : t > 0\}$, so the family becomes to a directed set when ordered by reverse inclusion. More precisely speaking, $\mathcal{F}$ is a time-dependent filter basis on the subsets of $\mathcal{S}$, that is, $\emptyset \notin \mathcal{F}$ and $\mathcal{S}_{\geq t+s} \subset \mathcal{S}_{\geq t} \cap \mathcal{S}_{\geq s}$ for every $t, s > 0$. As an assumption for a control range, $\mathrm{cls}(\mathcal{S}_{\leq t}(x))$ and $\mathrm{cls}(\mathcal{S}_{\leq t}^{-1}(x))$ are also compact for every $t > 0$ and $x \in M$(see [7] for details).

## 2. Main result

Throughout this section, let $M$ be a closed connected smooth manifold with metric $d$. We denote by $\mathcal{C}(M)$ the set of nonempty closed subsets of the space $M$. Note that every closed subset of $M$ is also compact as well. We consider the space $\mathcal{C}(M)$ with the topology induced by

the Hausdorff metric. For a multi-valued mapping $f : M \to \mathcal{C}(M)$, let us define the continuity of the mapping $f$.

For $x \in M$, we say that the mapping $f$ is *upper semicontinuous* at $x$ if for any number $\varepsilon > 0$ there exists a neighborhood $U$ of $x$ in $M$ such that

$$f(y) \subset B(f(x), \varepsilon), \ y \in U.$$

Here, $B(f(x), \varepsilon)$ is an open $\varepsilon$-ball of the compact set $f(x)$.

For $x \in M$, we say that the mapping $f$ is *lower semicontinuous* at $x$ if for any number $\varepsilon > 0$ there exists a neighborhood $U$ of $x$ in $M$ such that

$$f(x) \subset B(f(y), \varepsilon), \ y \in U.$$

We say that the mapping $f$ is upper semicontinuous [lower semicontinuous continuous] if $f$ is upper semicontinuous [lower semicontinuous continuous] at every point on M. We say that the mapping $f$ is *continuous* if and only if $f$ is both upper and lower semicontinuous.

Given a control system $\sum$, we give the definition of an important multi-valued mapping $J^+$ from $M$ into the family $\mathcal{C}(M)$ defined by $y \in J^+(x)$ if and only if there are sequences $\{t_n\}$ of $\mathbb{R}^+$, $\{x_n\}$ of $M$ and $\{u_n\}$ of $\mathcal{U}_{pc}$ satisfying the property that $t_n \to \infty$, $x_n \to x$ and $\phi(t_n, x_n, u_n) \to y$. We call the set $J^+(x)$ as *first positive prolongational limit set* of $x \in M$. It is easily proved that the first positive prolongational limit set $J^+(x)$ of $x$ is expressed by

$$J^+(x) = \bigcap_{t>0} \bigcap_{\varepsilon>0} \text{cls}(\mathcal{S}_{\geq t} B(x, \varepsilon)).$$

In the following theorem, we mention upper semicontinuity of the multi-valued mapping $J^+$.

THEOREM 2.1. *The first positive prolongational limit multi-valued mapping $J^+ : M \to \mathcal{C}(M)$ is upper semicontinuous on the whole space $M$.*

*Proof.* Let $x \in M$ and $\varepsilon > 0$. Now we will prove that there exist $\delta > 0, \tau > 0$ such that $\mathcal{S}_{\geq \tau}(B(x, \delta)) \subseteq B(J^+(x), \frac{\varepsilon}{2})$. To contrary, suppose that for any $r > 0$ and $t > 0$, $\mathcal{S}_{\geq t}(B(x, r)) \not\subseteq B(J^+(x), \frac{\varepsilon}{2})$. Then for every positive integer $n$, we have $\mathcal{S}_{\geq n}(B(x, \frac{1}{n})) \not\subseteq B(J^+(x), \frac{\varepsilon}{2})$. So there exist $t_n \geq n$, $x_n \in B(x, \frac{1}{n})$ and $u_n \in \mathcal{U}_{pc}$ such that $\phi(t_n, x_n, u_n) \notin B(J^+(x), \frac{\varepsilon}{2})$. By compactness of $M$, the sequence $\{\phi(t_n, x_n, u_n)\}$ has a subsequence that converges to some point in $M$. Without loss of generality we may assume that the sequence $\{\phi(t_n, x_n, u_n)\}$ converges to a point $p$ in $M$. Since $t_n \to \infty$ and $x_n \to x$, we obtain that $p \in J^+(x)$.

On the other hand, because of $\phi(t_n, x_n, u_n) \notin B(J^+(x), \frac{\varepsilon}{2})$, we get that $p \notin B(J^+(x), \frac{\varepsilon}{2})$ so we come to a contradiction. Therefore there exist $\delta > 0$, $\tau > 0$ such that $\mathcal{S}_{\geq \tau}(B(x, \delta)) \subseteq B(J^+(x), \frac{\varepsilon}{2})$.

For $y \in B(x, \delta)$, we let $\eta := \delta - d(x, y)$. Then we get immediately that $\eta > 0$ and $B(y, \eta) \subseteq B(x, \delta)$. Therefore we obtain the following inclusions,

$$
\begin{aligned}
J^+(y) &\subseteq \operatorname{cls}(\mathcal{S}_{\geq \tau}(B(y, \eta))) \\
&\subseteq \operatorname{cls}(\mathcal{S}_{\geq \tau}(B(x, \delta))) \\
&\subseteq \operatorname{cls}(B(J^+(x), \frac{\varepsilon}{2})) \\
&\subseteq B(J^+(x), \varepsilon).
\end{aligned}
$$

Hence $J^+$ is upper semicontinuous at $x$, as required. $\qquad \square$

REMARK 2.2. According to results in [6], it is obtained that $J^+$ is also lower semicontinuous. Thus we conclude that $J^+$ is continuous on a closed connected smooth manifold $M$ under the given control system.

Now, we focus on another important multi-valued mapping $D^+$. We define the multi-valued mapping $D^+$ from $M$ into the family $\mathcal{C}(M)$ given by $y \in D^+(x)$ if and only if there are sequences $\{t_n\}$ of $\mathbb{R}^+$, $\{x_n\}$ of $M$ and $\{u_n\}$ of $\mathcal{U}_{pc}$ satisfying the property that $x_n \to x$ and $\phi(t_n, x_n, u_n) \to y$. The set $D^+(x)$ is called the *first positive prolongation* of $x \in M$. From the definition, we easily obtain

$$
D^+(x) = \bigcap_{\varepsilon > 0} \operatorname{cls}(\mathcal{S}(B(x, \varepsilon))).
$$

THEOREM 2.3. *The first positive prolongation mapping $D^+ : M \to \mathcal{C}(M)$ is upper semicontinuou.*

*Proof.* Let $x \in M$ and $\varepsilon > 0$. Then it is easily checked that

$$
\operatorname{cls}(B(D^+(x), \frac{\varepsilon}{2})) \subseteq B(D^+(x), \varepsilon).
$$

We claim that there exists a positive real number $\delta$ such that

$$
\mathcal{S}(B(x, \delta)) \subseteq B(D^+(x), \frac{\varepsilon}{2}).
$$

On the contrary, suppose that $\mathcal{S}(B(x, \frac{1}{n})) \not\subseteq B(D^+(x), \frac{\varepsilon}{2})$ for all $n \in \mathbb{N}$. Thus there exist $x_n \in B(x, \frac{1}{n})$, $t_n \geq 0$ and $u_n \in \mathcal{U}_{pc}$ such that $\phi(t_n, x_n, u_n) \notin B(D^+(x), \frac{\varepsilon}{2})$. From the compactness of $M$, the sequence $\{\phi(t_n, x_n, u_n)\}$ has a convergent subsequence. So, without loss of generality, we can pick up a point $p \in M$ such that $\phi(t_n, x_n, u_n) \to p$.

Since $x_n \to x$, we have $p \in D^+(x) \subseteq B(D^+(x), \frac{\varepsilon}{2})$. Since $\phi(t_n, x_n, u_n) \notin B(D^+(x), \frac{\varepsilon}{2})$, we get $p \notin B(D^+(x), \frac{\varepsilon}{2})$ which is a contradiction. Therefore there exists $\delta > 0$ such that $S(B(x, \delta)) \subseteq B(D^+(x), \frac{\varepsilon}{2})$. For every $y \in B(x, \delta)$ set $\eta := \delta - d(x, y)$, then $\eta > 0$ and $B(y, \eta) \subseteq B(x, \delta)$. Thus it follows that $D^+(y) \subseteq B(D^+(x), \varepsilon)$ using the similar method in Theorem 2.1. Therefore $D^+$ is upper semicontinuous at $x$ which completes the proof. $\qquad\square$

REMARK 2.4. For the case of the first positive prolongation mapping $D^+$, we also have that $D^+$ is lower semicontinuous by the results in [6]. Thus we also conclude that $D^+$ is continuous on a closed connected smooth manifold.

# References

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